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Battle-Outcome Prediction for an Extended System of Lanchester-Type Differential Equations

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Battle-outcome-prediction conditions are given for an extended system of Lanchester-type differential equations for two different types of battle-termination conditions: (a) fixed-force-level-breakpoint battles, and (b) fixed-force-ratio-breakpoint battles. Necessary and sufficient conditions for predicting battle outcome are given in the former case for a fight to the finish, while sufficient conditions are given in the latter case. The former results are equivalent to those for the problem of classical analysis of determining (explicitly as a function of the initial conditions) the occurrence of a zero point for the solution to this extended system, although such results as given here have not appeared proviously for nonoscillatory (in the strict sense) solutions.

1. INTRODUCTION

Lanchester-type combat models are widely used in military operations-research (OR) activities [1, 3, 7, 15]. This paper studies the qualitative behavior of solutions to the following extended system of Lanchester-type differential equations for $0 \leqslant t \leqslant t_f$

$$\begin{aligned} \frac{dx}{dt} &= -a(t)\,y - \beta(t)\,x \qquad \text{with} \quad x(0) = x_0, \\ \frac{dy}{dt} &= -b(t)\,x - \alpha(t)\,y \qquad \text{with} \quad y(0) = y_0, \end{aligned} \tag{1.1}$$

where $t = 0$ denotes the time at which the battle begins, t_f denotes the time at which it ends, $x(t)$ and $y(t)$ denote the numbers of X and Y at time t, and $a(t)$, $b(t)$, $\alpha(t)$, and $\beta(t)$ denote (nonnegative) time-dependent Lanchester attrition-rate coefficients, which represent the effectiveness of each side's fire. On physical grounds we must have $x(t)$ and $y(t) > 0$ for $0 < t < t_f$. One

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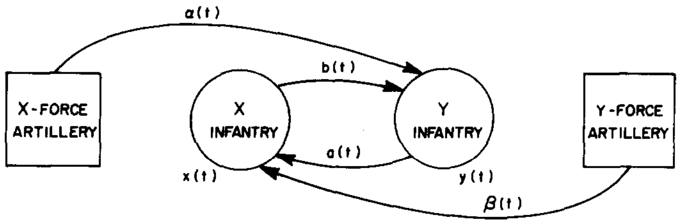

FIG. 1. Operational situation modelled by extended system of Lanchester-type differential equations: combat between two homogeneous primary forces (infantries) with superimposed effects of supporting weapons (artillery) not subject to attrition.

important operational interpretation [14] of the model (1.1) is that of representing "aimed-fire" combat between two homogeneous (primary) forces with superimposed effects of supporting fires not subject to attrition (see Fig. 1).

Battle-outcome prediction (i.e., the prediction of which one of two or more mutually exclusive terminal sets will be reached) is an important problem of OR [7, 9, 12]. It leads not only to problems of classical analysis but also to some interesting nonstandard problems whose statement is most conveniently given in the terminology of Lanchester combat theory. In particular, Theorems 2 and 3 are not standard ones (cf. our work on the real zeros of nonoscillatory (in the strict sense) solutions to linear second-order equations [8, 12] and error bounds for the Liouville–Green approximation [6, 10]).

The modelling of battle termination involves the concept of a military unit's "breakpoint": that point beyond which the unit cannot continue to fight and carry out its mission and consequently seeks to "break off" the engagement (see [9] for further details). Two widely used deterministic models of battle termination are (I) battle terminated by one side's force level reaching a given value (called the unit's "breakpoint" force level) while the other side's force level has always been above its breakpoint value (*fixed-force-level-breakpoint battle*), and (II) battle terminated by the force ratio first reaching either one of two given "breakpoint" values (*fixed-force-ratio-breakpoint battle*). We will consider only the special case of the former in which the forces fight until one or the other is annihilated. In the latter case we introduce the force ratio $z = x/y$ and denote X's breakpoint force ratio as z_{BP}^X (with z_{BP}^Y being similarly defined). These breakpoint force ratios satisfy $0 \leqslant z_{BP}^X < z_0 = z(0) < z_{BP}^Y \leqslant +\infty$. It follows that, for example, Y wins such a fixed-force-ratio-breakpoint battle at time t_f when (1) $z(t_f) = z_{BP}^X$, and (2) $0 \leqslant z_{BP}^X < z(t) < z_{BP}^Y \leqslant +\infty$ for $0 \leqslant t < t_f$. Corresponding to a fight until the annihilation of one side or the other is the case in which $z_{BP}^X = 0$ and $z_{BP}^Y = +\infty$.

Work on battle-outcome-prediction conditions for Lanchester-type differential equations was initiated by Taylor and Parry [14] in 1975 and subsequently pursued in [8, 9, 12, 13] (usually for (1.1) with $\alpha(t) \equiv \beta(t) \equiv 0$). In [8] the author observed that the only known work on determining the (at most) single zero on the real line of the general linear second-order differential equation with nonoscillatory (in the strict sense) solutions appears in Hille's book [2, Sect. 9.2]. However, the work of Shreve [5] and Petty and Johnson [4] is related in spirit to the work at hand.

2. Preliminaries

Mathematically, we make the following assumptions about the attrition-rate coefficients:

(A1) $a(t)$, $b(t)$, $\alpha(t)$, and $\beta(t) \geqslant 0$ for all $t \geqslant t_1$ with $t_1 \leqslant 0$,

(A2) $a(t)$, $b(t)$, $\alpha(t)$, and $\beta(t) \in L(t_1, T)$ for any finite $T \geqslant t_1$,

where t_1 equals either 0 or $t_0 \leqslant 0$, and t_0 denotes a conveniently chosen initial point for a certain related initial-value problem (see [8] and Sect. 3 below) that defines the canonical functions used to represent solutions to (1.1). The choice of t_0 subsumes that $a(t)$ and $b(t)$ are defined, positive, and absolutely continuous for all $t > t_0$. We will take (cf. [8, 13]) $a(t)$ and $b(t)$ to be given in the form $a(t) = k_a g(t)$ and $b(t) = k_b h(t)$, where k_a and k_b are positive constants chosen so that $R(t) \equiv k_a/k_b$ if and only if $g(t) \equiv h(t)$. It is then convenient [7, 8] to introduce the *relative-fire-effectiveness parameter* λ_R defined by $\lambda_R = k_a/k_b$. We also let

$$t_* = \inf \left\{ \xi \;\middle|\; \begin{array}{l} a(t) \text{ and } b(t) \text{ are positive, absolutely continuous} \\ \text{real-valued functions defined on all compact} \\ \text{subsets of } [\xi, +\infty) \end{array} \right\}$$

and assume that $t_* \leqslant 0$. Moreover, all solutions to (1.1) are nonoscillatory in the strict sense, and this result does not depend on the signs of $\alpha(t)$ and $\beta(t)$.

Theorem 1. *Assume that $a(t)$ and $b(t) \geqslant 0$ for all $t \geqslant 0$ and that* (A2) *holds with $t_1 = 0$. Let $x(t)$ and $y(t)$ satisfy* (1.1). *Then between them both $x(t)$ and $y(t)$ can have at most one zero for all finite $t \geqslant 0$, and each can have at most one zero for all $t \geqslant 0$.*

Proof. Multiply the first of equations (1.1) by y, the second by x, add, rearrange, and integrate to obtain

$$x(t)\,y(t) = \left\{\exp\left[-\int_0^t \{\alpha(s)+\beta(s)+2\sqrt{a(s)\,b(s)}\}\,ds\right]\right\}$$
$$\times\left\{x_0 y_0 - \int_0^t \{\sqrt{a(s)}\,y(s)-\sqrt{b(s)}\,x(s)\}^2\right.$$
$$\left.\times\exp\left[\int_0^s \{\alpha(r)+\beta(r)+2\sqrt{a(r)\,b(r)}\}\,dr\right]ds\right\},$$

whence follows the theorem, since the simultaneous vanishing of both $x(t)$ and $y(t)$ at any finite t is precluded by the uniqueness of solutions to (1.1).

Q.E.D.

The force ratio $z = x/y$ satisfied the Riccati equation

$$\frac{dz}{dt} = b(t)\,z^2 + \{\alpha(t)-\beta(t)\}\,z - a(t) \qquad \text{with} \quad z(0) = x_0/y_0, \tag{2.1}$$

which is also conveniently written as

$$\frac{dz}{dt} = b(t)\{z^2 - \sqrt{R(t)}\,S(t)\,z - R(t)\}, \tag{2.2}$$

where

$$R(t) = \frac{a(t)}{b(t)} \qquad \text{and} \qquad S(t) = \frac{\beta(t)-\alpha(t)}{\sqrt{a(t)\,b(b)}}. \tag{2.3}$$

Denote the two zeros of the quadratic expression in (2.2) as $z_{\pm}^*(t)$ and observe that $z_-^*(t) \leqslant 0 \leqslant z_+^*(t) = \sqrt{R(t)}\{S(t)/2 + \sqrt{[S(t)/2]^2+1}\}$. It follows from (2.2) that when $a(t)$ and $b(t) > 0$, $dz/dt(t) < 0$ for $z_-^*(t) < z(t) < z_+^*(t)$. Denote $R(0)$ as R_0 and $S(0)$ as S_0. It is then easy to prove the following two lemmas.

LEMMA 1. *If $R(t) \geqslant R_0$ and $S(t) \geqslant S_0$ for all $t \geqslant 0$, then $z_+^*(t) \geqslant z_+^*(0)$.*

LEMMA 2. *If $dz/dt(0) < 0$ and $z_+^*(t) \geqslant z_+^*(0)$, then $dz/st(t) \leqslant 0$ for all $t \geqslant 0$, with strict inequality holding when $b(t) > 0$.*

3. Prediction of Zero Points for the Extended System

The substitution $p(t) = x(t)\exp\{\int_0^t \beta(s)\,ds\}$, $q(t) = y(t)\exp\{\int_0^t \alpha(s)\,ds\}$ transforms (1.1) into

$$\begin{aligned} \frac{dp}{dt} &= -A(t)\,q \qquad \text{with} \quad p(0) = x_0, \\ \frac{dq}{dt} &= -B(t)\,p \qquad \text{with} \quad q(0) = y_0, \end{aligned} \tag{3.1}$$

where $A(t) = a(t)\exp\{\int_0^t [\beta(s) - \alpha(s)]\,ds\}$ and $B(t) = b(t)\exp\{-\int_0^t [\beta(s) - \alpha(s)]\,ds\}$. The transformed "force-level" variable $p(t)$ satisfies $d^2p/dt^2 - \{[1/A(t)]\,dA/dt\}\,dp/dt - A(t)\,B(t)\,p = 0$, which may be written in the equivalent form

$$\frac{d^2p}{dt^2} - \left\{\beta(t) - \alpha(t) + \frac{1}{a(t)}\frac{da}{dt}\right\}\frac{dp}{dt} - a(t)\,b(t)\,p = 0, \tag{3.2}$$

with initial conditions $p(0) = x_0$ and $\{1/a(0)\}\,dp/dt(0) = -y_0$. Hence (for $t_* \leqslant 0$), by the results of [7, 8]

$$\begin{aligned} p(t) = x_0\{&C_Q(0)\,C_P(t) - S_Q(0)\,S_P(t)\} \\ &- y_0\sqrt{\lambda_R}\{C_P(0)\,S_P(t) - S_P(0\,C_P(t)\}, \end{aligned} \tag{3.3}$$

where the hyperbolic-like GLF $C_P(t)$ and $S_p(t)$ are linearly independent solutions to the P force-level equation (3.2) that satisfy the initial conditions $C_P(t_0) = 1$, $\{1/a(t_0)\}\,dC_P/dt(t_0) = 0$, $S_P(t_0) = 1$, and $\{1/a(t_0)\}\,dS_P/dt(t_0) = 1/\sqrt{\lambda_R}$, with $t_0 \in [t_*, 0]$. It has been found convenient [8, 11] to take either $t_0 = t_* \leqslant 0$ or $t_0 = 0$. The GLF $C_Q(t)$ and $S_Q(t)$ are similarly defined, with the following initial condition worthy of note: $\{1/b(t_0)\} \cdot dS_Q/dt(t_0) = \sqrt{\lambda_R}$. It follows that the X force level to (1.1) may be written as

$$\begin{aligned} x(t) = [&x_0\{C_Q(0)\,C_P(t) - S_Q(0)\,S_P(t)\} \\ &- y_0\sqrt{\lambda_R}\{C_P(0)\,S_P(t) - S_P(0)\,C_P(t)\}]\exp\left\{-\int_0^t \beta(s)\,ds\right\}. \end{aligned} \tag{3.4}$$

Since $p(t)$ and $x(t)$ have te same finite zero points, we can invoke Theorem 2 of [8] to conclude the following theorem.

THEOREM 2. *Assume that* (A1) *and* (A2) *hold with* $t_1 = t_0$. *Then the* X *force level* $x(t)$ *to the extended system* (1.1) *has a finite zero point if and only if*

$$z_0 < \sqrt{\lambda_R}\,G(\Lambda^*_{\max}), \tag{3.5}$$

where $G(\Lambda)$ *is given by* $G(\Lambda) = \{C_P(0) - \Lambda S_P(0)\}/\{\Lambda C_Q(0) - S_Q(0)\}$ *and is* *a strictly decreasing function of its argument Λ. Neither side will be annihilated in finite time if and only if*

$$\sqrt{\lambda_R}\, G(\Lambda^*_{\max}) \leqslant z_0 \leqslant \sqrt{\lambda_R}\, G(\Lambda^*_{\min}), \tag{3.6}$$

where

$$\lim_{t\to+\infty} \frac{S_P(t)}{C_P(t)} = \frac{1}{\Lambda^*_{\max}} = \frac{1}{\sqrt{\lambda_R}} \int_{t_0}^{\infty} \frac{A(s)\, ds}{\{C_P(s)\}^2}, \tag{3.7}$$

and

$$\lim_{t\to+\infty} \frac{S_Q(t)}{C_Q(t)} = \Lambda^*_{\min} = \sqrt{\lambda_R} \int_{t_0}^{\infty} \frac{B(s)\, ds}{\{C_Q(s)\}^2}. \tag{3.8}$$

We always have $0 < \Lambda^*_{\min} \leqslant \Lambda^*_{\max} < +\infty$, with $\Lambda^*_{\min} < \Lambda^*_{\max}$ if and only if both $A(t)$ and $B(t) \in L(t_0, +\infty)$.

Remark 1. Both $A(t)$ and $B(t) \in L(t_0, +\infty)$ implies by the Cauchy–Schwarz inequality that $\sqrt{a(t)\, b(t)} \in L(t_0, +\infty)$. Here $\sqrt{a(t)\, b(t)}$ may be operationally interpreted as the intensity of the primary-force combat [7, 14].

4. Sufficient Conditions in a Special Case

Let us now additionally assume

(A3) $R(t) \geqslant R_0$ and $S(t) \geqslant S_0$, and

(A4) $b(t) \notin L(0, +\infty)$.

Theorem 3. *Consider* (1.1) *and assume that* (A1) *through* (A4) *hold with $t_1 = 0$. Then*

$$z_0 < \sqrt{R_0} \left\{ \frac{S_0}{2} + \sqrt{\left(\frac{S_0}{2}\right)^2 + 1} \right\} \tag{4.1}$$

implies that X will lose any fixed-force-ratio-breakpoint battle in finite time.

Proof. It follows from Lemmas 1 and 2 that (4.1) implies that $dz/dt(t) \leqslant 0$ for all $t \geqslant 0$, with strict inequality holding when $b(t) > 0$. It remains to show that $z(t) \to z^X_{BP} \in [0, z_0)$ in finite time. The theorem will be proven by showing that

$$z(t) \leqslant z_0 - C \int_0^t b(s)\, ds \tag{4.2}$$

with $C > 0$, since $b(t) \notin L(0, +\infty)$ then implies that $z(t) \to 0$ in finite time. It suffices to consider the case in which $R_0 > 0$. When $S_0 \leqslant 0$, $dz/dt = b(t) R(t)\{z^2/R(t) + [-S(t)/\sqrt{R(t)}]\ z - 1\} \leqslant b(t)\ R(t)\{z_0^2/R_0 + (-S_0/\sqrt{R_0})\ z_0 - 1\} \leqslant b(t)\ R_0\{z_0^2/R_0 + (-S_0/\sqrt{R_0})\ z_0 - 1\}$, whence (4.2) holds with $C = -(1/b_0)\, dz/dt(0)$. When $S_0 \geqslant 0$, there are two cases to be considered. For $0 \leqslant z \leqslant \sqrt{R(t)}\ S(t)/2$, $dz/dt \leqslant -a(t) \leqslant -b(t) R_0$. For $\sqrt{R(t)}\ \ S(t)/2 \leqslant z < z_+^*(t)$, $dz/dt = b(t)\ R(t)\{z/\sqrt{R(t)} - S(t)/2]^2 - (1 + [S(t)/2]^2)\} \leqslant \mathrm{b(t)}\ R(t)\{(z_0/\sqrt{R_0} - S_0/2)^2 - [1 + (S_0/2)^2]\} \leqslant b(t)\ R_0\{(z_0/\sqrt{R_0} - S_0/2)^2 - [1 + (S_0/2)^2]\}$. Hence, (4.2) again holds with $C = \min(R_0, -(1/b_0)(dz/dt)(0))$. Q.E.D.

5. Examples

Here we consider a couple of examples that highlight the role of coefficient integrability in battle-outcome-prediction conditions (i.e., Theorems 2 and 3).

Example 1. Consider (cf. [14, Sect. 3]) $a(t) = k_a h(t)$, $b(t) = k_b h(t)$, and $\alpha(t) \equiv \beta(t)$, and take $t_1 = t_0 = 0$. It follows that $1/\Lambda^*_{\max} = (1 - e^{-2M})/(1 + e^{-2M}) = \Lambda^*_{\min} \leqslant 1$, where $M = \lim_{t \to +\infty} \sqrt{k_a k_b} \cdot \int_0^t h(s)\, ds$. Neither side will be annihilated in finite time for

$$\sqrt{\lambda_R}\ \left(\frac{1 - e^{-2M}}{1 + e^{-2M}}\right) \leqslant z_0 \leqslant \sqrt{\lambda_R}\ \left(\frac{1 + e^{-2M}}{1 - e^{-2M}}\right), \tag{5.1}$$

and hence when $M < +\infty$, $z_0 < \sqrt{\lambda_R}$ does not imply that X will lose a fixed-force-ratio-breakpoint battle in finite time (even though $dz/dt(t) < 0$ for all $t > 0$), since condition (A4) of Theorem 3 is not satisfied.

Example 2. Consider (cf. [12]) $a(t) = k_a e^{\lambda_a t}$, $b(t) = k_b e^{\lambda_b t}$, $\alpha(t) = K_\alpha$, and $\beta(t) = K_\beta$. In this case $R(t) = R_0 e^{(\lambda_a - \lambda_b)t}$, $R_0 = \lambda_R$, $S(t) = S_0 e^{-(\lambda_a + \lambda_b)t/2}$, and $S_0 = (K_\beta - K_\alpha)/\sqrt{k_a k_b}$. We will assume that $R(t) \geqslant R_0$ and $S(t) \geqslant S_0$. Let us observe that if both λ_a and $\lambda_b \geqslant 0$, then both $a(t)$ and $b(t) \notin L(0, +\infty)$; while if both λ_a and $\lambda_b < 0$, then both $a(t)$ and $b(t) \in L(0, +\infty)$. We will cnsider three cases.

Case (a): *Both* λ_a *and* $\lambda_b > 0$. In this case $C_P(t) = F_q(\tau)$, $S_P(t) = (\tau_0/2)^{1-2p} H_p(\tau)$, $C_Q(t) = F_p(\tau)$, and $S_Q(t) = (\tau_0/2)^{2p-1} H_q(\tau)$, where $\tau(t) = \tau_0 e^{(\lambda_a + \lambda_b)t/2}$, $\tau_0 = 2\sqrt{k_a k_b}/(\lambda_a + \lambda_b)$, $p = (\lambda_a - K_\alpha + K_\beta)/(\lambda_a + \lambda_b)$, $q = 1 - p$, and we assume that p is not equal to an integer or zero. Here $F_\alpha(\xi)$ and $H_\alpha(\xi)$ denote so-called Lanchester–Clifford–Schläfli (LCS) functions [13], which may be represented for $\alpha \neq 0, -1, -2, \ldots$, as $F_\alpha(\xi) = \Gamma(\alpha)$

$\sum_{k=0}^{\infty} \{(\xi/2)^{2k}/[k!\Gamma(k+\alpha)]\}$ and $H_\alpha(\xi) = \Gamma(\alpha) \sum_{k=0}^{\infty} \{(\xi/2)^{2(k+\alpha)}/[k!\Gamma(k+\alpha+1)]\}$. It follows that [8] $\Lambda^*_{\max} = \Lambda^*_{\min} = \Lambda^* = (\tau_0/2)^{2p-1} \Gamma(1-p)/\Gamma(p)$. Thus, X will be annihilated in finite time if and only if

$$z_0 < \sqrt{\lambda_R} \left(\frac{\tau_0}{2}\right)^{1-2p} \left\{\frac{F_q(\tau_0)\,\Gamma(p) - H_p(\tau_0)\,\Gamma(q)}{F_p(\tau_0)\,\Gamma(q) - H_q(\tau_0)\,\Gamma(p)}\right\}. \tag{5.2}$$

However, when $\lambda_a \geqslant \lambda_b$ and $K_\alpha \geqslant K_\beta$, then simple (i.e., not involving any transcendental functions) sufficient condition (4.1) reads

$$z_0 < \sqrt{\lambda_R} \left\{\left(\frac{K_\beta - K_\alpha}{2\sqrt{k_a k_b}}\right) + \sqrt{\left(\frac{K_\beta - K_\alpha}{2\sqrt{k_a k_b}}\right)^2 + 1}\right\} \tag{5.3}$$

and implies the more general outcome that X will lose any fixed-force-ratio-breakpoint battle in finite time.

Case (b): *Both* λ_a *and* $\lambda_b = 0$. It follows that $\Lambda^* = \sqrt{1 + (S/2)^2} - S/2$, where $S = (K_\beta - K_\alpha)/\sqrt{k_a k_b}$. In this case (5.3) is both necessary as well as sufficient for X to lose any fixed-force-ratio-breakpoint battle in finite time.

Case (c): *Both* λ_a *and* $\lambda_b < 0$. In this case it is more convenient to abandon the basic paradigm of Section 3. Focusing on $x(t)$ and $y(t)$, one finds by analysis similar to that employed in [8] that neither side will be annihilated in finite time if and only if

$$\sqrt{\lambda_R} \left(\frac{\tau_0}{2}\right)^{1-2p} \frac{H_p(\tau_0)}{F_p(\tau_0)} \leqslant z_0 \leqslant \sqrt{\lambda_R} \left(\frac{\tau_0}{2}\right)^{1-2p} \frac{F_q(\tau_0)}{H_q(\tau_0)}, \tag{5.4}$$

where $p = (\lambda_a - K_\alpha + K_\beta)/(\lambda_a + \lambda_b)$, $q = 1 - p$, and $\tau_0 = 2\sqrt{k_a k_b}/(-\lambda_a - \lambda_b)$. Here the length of the initial-force-ratio interval for which neither side will be annihilated in finite time is given by $\sqrt{\lambda_R}(\tau_0/2)^{1-2p}/\{F_p(t_0)\,H_q(\tau_0)\} > 0$. Moreover, when $\lambda_a \geqslant \lambda_b$ and $K_\alpha \leqslant K_\beta$, (5.3) does not imply that X will lose a fixed-force-ratio-breakpoint battle in finite time, since the battle may never reach the termination condition $z_f = z^X_{BP}$. The reader will find it instructive to work out such details for $\lambda_a - K_\alpha + K_\beta = \lambda_b + K_\alpha - K_\beta$ (i.e., $p = 1/2$).

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